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Design of a tunable detector for gravitational radiation†

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Abstract. A design is proposed for a cryogenic detector for gravitational radiation. With a proper choice of a set of design parameters, the resonant frequency of the detector can be pre-selected to lie within 10^{-3} to 10^{3} Hz; calculated values of these parameters are tabulated. Fine tuning of the detector is also possible. Using SQUID magnetometers to measure displacement, the expected gravitational radiation flux sensitivity is shown to be between 10^{4} and 10^{8} J m^{-2} per cycle for burst radiation, and less than 10^{-4} W m^{-2} for continuous radiation at pulsar frequencies.

1. Introduction

Recent technological advances have generated interest in experimental tests of general relativity. Much of the current inquiry is focused on the possibility of detecting gravitational radiation from astrophysical sources.

Since Weber's work in the sixties, several papers have reviewed the development of gravitational radiation detectors (Braginskii and Rodenko 1970, Press and Thorne 1972, Drever 1977). Reports of theoretical work on possible sources (Press and Thorne 1972, Drever 1977) and detection schemes, as well as experimental work on sensitive transducers (Paik 1976, Adami *et al* 1977, Hoffman *et al* 1976, Blair *et al* 1977, Kos and Ramadan 1977), have also been published.

Although low-frequency detectors have been operated by some groups (Forward 1978, Hirakawa and Narihara 1975, Hirakawa and Tsubono 1978), most antennae for gravitational radiation, whether operational or under construction, have been designed to detect radiation in the kilohertz frequency range. It is believed that collapsing stars would trigger such detectors. Radiation from other astrophysical sources spans a wide range of frequencies. Some of these sources emit radiation which is monochromatic, others emit broad-band bursts. Radiation from a few of these sources is also polarised.

Among other desirable features, such as low sensitivity, a useful detector should also be sufficiently versatile to accommodate radiation from a variety of sources. This would demand a detector that is tunable and is sensitive to the polarisation characteristics of the incident radiation as well. The detector design given here meets some of these requirements.

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2. Design

2.1. Physical description

Inside an evacuated cylindrical cavity are located two identical cylinders. The cavity and cylinders are made of materials which are in a superconducting state. Both cylinders carry persistent currents labelled I_2 which are identical and axially symmetric. (A similar device has been proposed as an impedance transformer between transducer and antenna by Lavrent'ev (1970).) Along the inner circumference and coaxial with the cavity is a tuning coil of radius a with N_1 turns carrying a persistent current I_1 in a sense opposite to I_2. The spatial relationship of these components and other details are given in figure 1.

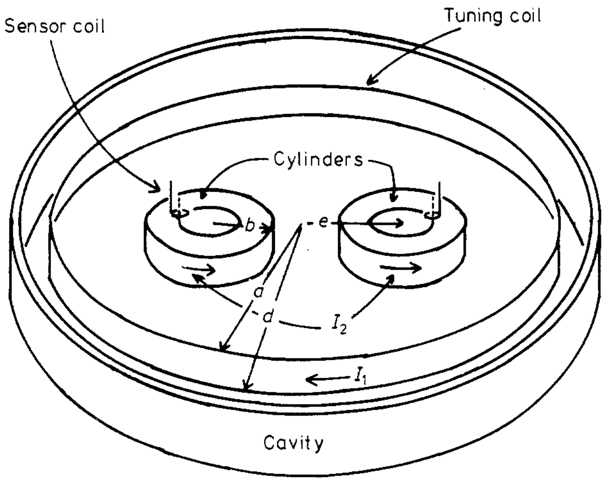

Figure 1. Isometric view of detector cavity showing spatial arrangement of levitating cylinders, sensor coils, tuning coil and direction of persistent currents. The cavity lid has been left out for the sake of clarity. Dimensions referred to in equation (2) are also indicated. The location of the sensor coils is appropriate for the detection of oscillations in the radial mode.

Magnetic fields associated with these currents generate levitating and restoring forces due to the Meissner effect. For instance, a cylinder of mass 10 kg may be levitated to a height of a few millimetres with 10^4 A-turns (Carelli *et al* 1976).

Current I_2 may lie in a range whose upper limit is determined by the critical field of the superconductor in question. Critical fields can be as high as 10^{-1} T for some type I superconductors. On the other hand, I_2 may not be less than that current necessary to levitate the cylinder to some minimum height $z_0 \approx 10^{-4}$ m. The permissable range of I_2 is estimated to be between 10^2 and 10^4 A for a niobium or niobium-alloy cylinder of mass 10 kg. This cylinder is expected to have a height $h = 0{\cdot}07$ m, inner radius $R = 0{\cdot}015$ m and outer radius $b = 0{\cdot}075$ m.

2.2. Derivation of oscillation frequencies

2.2.1. Radial oscillation mode. For small radial displacements the motion of these cylinders is simple harmonic. Oscillation frequencies may be calculated using the spring–mass analogue of figure 2. The lowest mode of even parity has a frequency

$$\omega_r^2 = (k_1 + k_2 + 2k_3)/m \equiv k/m, \quad (1)$$

where m is the mass of each cylinder, and the spring constants are

$$k_1 = \mu_0 \pi I_2^2 b^4/d^5, \qquad k_2 = \mu_0 \pi N_1 I_1 I_2 R^2/a^3, \qquad k_3 = \mu_0 \pi I_2^2 b^4/(e+d)^5. \quad (2)$$

Dimensions a, b, e and d are indicated in figure 1, and $\mu_0 = 4\pi \times 10^{-7}$ H m^{-1}. Note that these springs transmit elastic forces at the velocity of light.

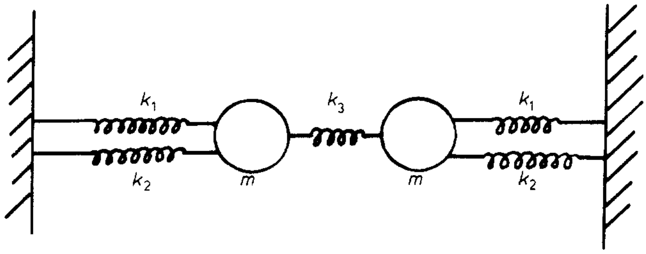

Figure 2. Spring–mass analogue of the oscillator. Spring constants k_1 and k_2 derive from the interaction of current loop I_2 with its image on the cavity wall and with the current loops $N_1 I_1$ of the tuning coil, respectively. The mutual interaction between the pair of current loops gives k_3.

Expressions for k_1 and k_3 were derived from the straightforward dipole–dipole interaction of a current loop of radius b with its image on the cavity wall, which is considered to be flat. Next we calculate terms in the series which represents the mutual inductance M of two co-planar current loops of radii R and a $(R < a)$, one inside the other but with centres offset by e. To obtain k_2 we take the second derivative of M with respect to the offset coordinate e, retaining only the first few terms, and multiply the result by $-N_1 I_1 I_2$ to obtain the final expression. There will be a numerical factor in equations (1) and (2) arising from (i) the circular geometry of the cavity wall from which images are reflected, (ii) the presence of the top and bottom surfaces of the cavity, and (iii) the vertical extent of the current loops. This factor, which has been left out, can be obtained by complicated numerical calculations which seem premature at this stage, since one is interested in order-of-magnitude estimates only.

2.2.2. Vertical oscillation mode. Vertical motion of the cylinders is also simple harmonic. The oscillation frequency may be calculated from the spring constant which is obtained from the mutual inductance of two closely spaced ($z_0 \ll b$) planar loops each carrying current I_2. This frequency may be changed by allowing a persistent current I_3 to flow in each of the two sensor coils which are positioned with their centres on the cylinder axes and parallel to the cavity lid.

The resultant frequency in the vertical mode is

$$\omega_v^2 = (\mu_0 R/m z_0^2)(I_2^2 + n I_3 I_2),$$

where n is the number of turns in the sensor coil.

Resonance frequencies for 10 kg cylinders oscillating within the cavity in the radial and vertical modes are listed against detector parameters in table 1.

Table 1. List of detector parameters required for various frequencies. Mass of each cylinder is 10 kg, and levitation height $z_0 = 10^{-4}$ *m*.

ω (s^{-1})	I_2 (A)	b/d	d (m)
Radial mode			
10^{-3}	10^2	0·1	1
10^{-2}	10^3	0·1	1
10^{-1}	10^4	0·1	1
1	10^4	0·5	0·25
Vertical mode			
10^2	10^2	0·1	1
10^3	10^3	0·1	1
10^4	10^4	0·1	1

2.3. *Tuning factor*

The tuning factor for the radial mode is

$$\mathrm{d}\omega_r/\mathrm{d}I_1 \approx 10^{-6} N_1 \text{ Hz-turn A}^{-1}$$

and for the vertical mode is

$$\mathrm{d}\omega_v/\mathrm{d}I_3 \approx (n/2)(1 + nI_3/I_2)^{-1/2} \text{ Hz A}^{-1}.$$

One is free to optimise this factor to suit experimental conditions. The tuning factor serves either to fine-tune the antenna or to scan the frequency of interest. Note also that the cavities are sufficiently compact ($d < 1$ m) that several of them can be stacked within a single dewar. If each cavity is designed for a different frequency, then we have in effect a detector that is sensitive to a range of discrete frequencies.

Frequencies accessible to this antenna correspond to gravitational radiation from sources such as binary stars, pulsars, and stars collapsing into black holes (Braginskii 1966, Misner *et al* 1973, Drever 1977).

2.4. *Dissipation mechanisms*

The Q factor depends on the ways energy is dissipated by the oscillator. Among these are the collisions of the cylinders with gas molecules in the cavity, bulk oscillations of the cylinders, and flexing of the cavity. The first of these mechanisms is negligible when the gas pressure inside the cavity is less than 10^{-7} Torr at 1 K. If the cavity and cylinders are made of niobium, which reportedly has a Q factor of 10^6 (Paik 1976) for $\omega = 10^3\ s^{-1}$, and the cavity is supported on quarter-wave posts at the nodal points, the overall Q factor will be $\approx 10^6$. Indeed, if Q increases with decreasing ω, as has been suggested for sapphire by Braginskii *et al* (1977), it is not unreasonable to expect a $Q \approx 10^8$ for $\omega \approx 10^2\ s^{-1}$.

The coil support deserves special attention. Because of the reaction suffered by the coil, a differential movement between the coil and its support is expected to dissipate energy; a fraction of this energy will also be transmitted to the support. A sensor coil that is sputtered around a support which has a high mechanical Q (such as single-crystal sapphire) will minimise this dissipation mechanism.

3. Detection method

3.1. Equivalent circuit

Small radial displacements of the cylinders are measured by monitoring flux changes in a superconductive sensor coil of n turns connected to a SQUID magnetometer.

The voltage v, current i, and displacement from equilibrium r that characterise the system are described by the equations

$$v = -n(\mathrm{d}\phi/\mathrm{d}r)\dot{r}, \tag{3}$$

$$m\ddot{r} + \lambda\dot{r} + kr = -(2\pi pnB_v)i \tag{4}$$

where $B_v = \mu_0 JRf(r)$ is the field at the centre of the sensor coil of radius p, which is at a distance r from the centre of one of the cylinder axes, and J is the average current density in the cylinder. The right-hand side of equation (4) represents the restoring force due to the relative motion of the sensor coil and the vertical component B_v of the induction field associated with each cylinder; all dissipative mechanisms are lumped into the decay constant λ. The function $f(r)$ describes the variation of B_v with r on a plane perpendicular to the cylinder axis. It was calculated by the method given in Brown and Flax (1964) and the result used to compute $\mathrm{d}\phi/\mathrm{d}r$, the flux gradient across the sensor coil (see equation (13)).

Solving equations (3) and (4) for v/i gives the impedance

$$Z = v/i = (\mathrm{j}\omega m/q + \lambda/q + k/\mathrm{j}\omega q)^{-1},$$

where $q = 2\pi pnB_v(n\ \mathrm{d}\phi/\mathrm{d}r)$ and j is the square root of -1. This relation indicates that the device can be represented by its equivalent circuit, which is a parallel combination of inductance L, capacitance C and resistance R' defined by

$$C = m/q, \qquad L = q/k, \qquad R' = q/\lambda. \tag{5}$$

The induced EMF across the sensor coil is amplified by a SQUID pre-amplifier which has an input inductance $L_s = 5\times10^{-7}$ H and energy sensitivity $E_m = 2\times10^{-30}$ J Hz^{-1}. (Other details of this pre-amplifier are described in Kos and Ramadan (1977)).

3.2. Displacement sensitivity

The amplitude of the noise current I_s in the pre-amplifier is determined by the relation

$$E_m = \tfrac{1}{2}I_s^2(L_s + L). \tag{6}$$

Since I_s is the minimum detectable current, the corresponding displacement of the cylinder $(\Delta r)_m$ is obtained from equations (3) and (4) as

$$(\Delta r)_m = I_s L_s (n\ \mathrm{d}\phi/\mathrm{d}r)^{-1}. \tag{7}$$

Noise injected into the antenna per cycle by the pre-amplifier is $I_s^2 R'/Q$.

Thermal excitations of energy $k_B T$ will drive the cylinders into oscillation with a mean amplitude

$$(\Delta r)_B = (4k_B T/m\omega^2)^{1/2} \tag{8}$$

(k_B is the Boltzmann constant, and T is the temperature of the system). If external signals are absent, the total mean amplitude of oscillation is

$$(\Delta r)_T = (4/m\omega^2)^{1/2}(k_B T + k_B T/Q + I_s^2 R'/Q)^{1/2}, \tag{9}$$

where the last two terms represent random excursions of the amplitude and are much smaller than the first term which is sinusoidal.

This calculation is similar to that in Kos and Ramadan (1977).

If gravitational radiation of energy E_g is absorbed by the oscillator, the total amplitude will increase to

$$(\Delta r) = (4/m\omega^2)^{1/2}(k_B T + E_g + k_B T/Q + I_s^2 R'/Q)^{1/2}. \tag{10}$$

It is clear that optimum sensitivity is achieved when

$$(\Delta r)_m = (\Delta r)_T - (\Delta r)_B.$$

This equation allows one to calculate n. From equations (8) and (9) one gets

$$(\Delta r)_m = \tfrac{1}{2}(\Delta r)_B(1/Q + I_s^2 R'/Qk_B T). \tag{11}$$

Substituting the expression for R' from equation (5) into equation (11) and using equation (7) gives an explicit relation for n:

$$I_s L_s(n\, d\phi/dr)^{-1} = \tfrac{1}{2}(\Delta r)_B[1/Q + I_s^2\,(2\pi p B_v)(d\phi/dr)n^2/k_B TQ\lambda], \tag{12}$$

where I_s is defined by equation (6). The term L in equation (6) also depends on n, as can be seen from equation (5). Thus

$$L = q/k = 2\pi p B_v(d\phi/dr)n^2/m\omega^2$$

and n is completely determined by a solution of equation (12).

3.3. Numerical estimates

3.3.1. Radial mode. The radial flux gradient $d\phi/dr$ can be written as

$$d\phi/dr = B_v\, dA/dr + A(d/dr)B_v, \tag{13}$$

where A is the area of that part of the sensor coil which lies in the region $0 < r < R$. The geometrical factor dA/dr is a function of the variables r, p and R. If the coil radius p is set equal to R, and the coil centre located close to the inner edge of the cylinder ($r \approx R$), then it can be shown that

$$|dA/dr| = |-1{\cdot}7R| = 2{\cdot}55 \times 10^{-2}\ m.$$

The current density $J = I_2/h(b-R)$, and the field gradient turns out to be $(dB_v/dr)_{r=R} = 10^{-4} I_2\ \mathrm{T\,m^{-1}}$. With $A \approx 10^{-4}\ \mathrm{m}^2$ and $10^2 < I_2 < 10^4\ A$, the flux gradient is $10^{-5} < (d\phi/dr)_{r=R} < 3 \times 10^{-3}\ \mathrm{Wb\,m^{-1}}$ and the range of B_v is $10^{-3} < B_v < 10^{-1}$ T. For $T = 1$ K one gets from equation (12)

$$5 < n < 79,$$

where the extrema correspond to $\nu = 10^{-3}$ Hz and $\nu = 1$ Hz respectively. The noise current calculated from eqation (6) is $10^{-13}\ \mathrm{A\,Hz^{-1/2}}$, and the minimum detectable displacement as obtained from equation (7) is

$$10^{-16} < (\Delta r)_m < 10^{-19}\ \mathrm{m}$$

within the same frequency interval. Both quantities n and $(\Delta r)_m$ are limited by thermal fluctuations ($1/2Q$ term in equation (12)).

3.3.2. Vertical mode. The vertical flux gradient relevant to vertical motion is

$$d\phi/dz = 2\pi p B_r,$$

where z is the vertical coordinate and B_r is the radial component of the induction field. When the sensor coil is placed symmetrically, the flux gradient falls in the range

$$10^{-4} < d\phi/dz < 10^{-2}\ \mathrm{Wb\ m^{-1}}$$

for frequencies between 10^2 and $10^3\ \mathrm{s^{-1}}$. One calculates the number of turns as described before and obtains

$$10^4 < n < 10^3,$$

for which the noise current $I_s \approx 10^{-12}\ \mathrm{A\ Hz^{-1/2}}$ and the corresponding displacement sensitivity is

$$10^{-18} < (\Delta z)_m < 10^{-19}\ \mathrm{m}$$

within the same frequency interval. At these frequencies SQUID noise exceeds thermal noise.

3.4. *Another detection method*

An alternative and perhaps more practical method of measuring radial displacements is to measure differences in amplitude of the modulated current in the tuning coils. Such a current will be generated by the relative motion of the cylinders (and the associated induction field) and the tuning coils. This scheme, where sensor coils are dispensed with, is applicable to radial modes only. Once again SQUIDS are used in measuring these currents.

Besides the simplicity of this arrangement, an added benefit comes from freeing the pair of cylinders to rotate about a vertical axis without affecting performance. This type of motion will occur if the detector is located away from the equator. The tuning range is improved too because the position of the sensor coil is irrelevant. Sensitivity figures are expected to remain unchanged if the number of turns in the tuning coil is optimised for maximum sensitivity.

However, sensor coils are necessary for the detection of vertical modes. In this case, azimuthal freedom is restricted by the persistent current I_3 in the sensor coils.

4. Energy sensitivity

Under ideal conditions a signal is detectable when the signal-to-noise ratio exceeds unity. If one applies this criterion, then the minimum detectable energy is obtained by setting

$$(\Delta r)_m = (\Delta r) - (\Delta r)_T;$$

hence

$$E_g \approx 2k_B T(\Delta r)_m/(\Delta r)_B.$$

This condition yields $E_g = 10^{-29}$ J per cycle for $\nu = 10^{-3}$ to 1 Hz (radial mode) and between $E_g = 10^{-27}$ and 10^{-26} J per cycle for $\nu = 10^2$ to 10^3 Hz (vertical mode). Under realistic conditions it ought to be feasible to detect energies of 10^{-28} and 10^{-26} J per cycle in the two frequency ranges.

5. Flux sensitivity

5.1. Burst radiation

Figures obtained for E_g may be converted to the gravitational radiation flux density ϵ through the Gibbons–Hawking (1971) equation. When the burst of radiation is of duration $2\pi/\omega$, one can write

$$\epsilon = (c^3/4Gm\pi^3)(2\pi/\omega)E_g/(2e)^2 \text{ J m}^{-2},$$

where c is the velocity of light and G is the gravitation constant.

The separation distance $2e$ varies from about 1 to 0·1 m for resonant frequencies between 10^{-3} and 1 s^{-1}. Using $E_g = 10^{-28}$ J per cycle one obtains

$$10^7 < \epsilon < 10^8 \text{ J m}^{-2}$$

per cycle for each polarisation mode. The corresponding figure for frequencies in the vertical mode is

$$10^5 < \epsilon < 10^4 \text{ J m}^{-2}$$

per cycle.

5.2. Monochromatic sources

Although the intensity of gravitational radiation incident on the earth from monochromatic sources (such as pulsars and binary stars) is abysmal, it is worth estimating the flux sensitivity of this detector for these sources.

For continuous sources the integration time can be increased to improve energy sensitivity. For sources such as the Vela pulsar ($\omega = 140 \text{ s}^{-1}$) the minimum detectable intensity I can be calculated from the expression (Pizzella 1975)

$$I = (\pi^3 c^3/32G)E_g/mQ^2(2e)^2 \text{ W m}^{-2}.$$

The number one obtains is $I = 10^{-4}$ W m^{-2}. If Q is greater than 10^6 at this frequency (say $Q \approx 10^8$), then $I = 10^{-8}$ W m^{-2}. This figure is interesting as it is close to the estimated upper limit of the intensity of gravitational radiation reaching the earth from the Vela pulsar (Zimmermann 1978).

6. Polarisation

The addition of two levitating cylinders to the existing ones in the cavity increases the cross section and also makes the detector more sensitive to the polarisation of the incident radiation. The four cylinders, located at the corners of a square each with its own sensor coil, have a number of normal modes, at least two of which couple with the oscillating components of the radiation field. A polarised wave will create a unique signature in the relative phase and amplitude of signals from the sensor coils.

7. Noise

Noise in the antenna is minimised in several ways. Because of its location the sensor coil is relatively insensitive to angular modes of oscillation, which have entirely different frequencies in any case. Furthermore, by connecting the two sensor coils symmetrically, contributions to the output signal from odd-parity modes are automatically eliminated. It may also be possible to reduce SQUID noise by operating a pair of SQUIDs (one for each sensor coil) in coincidence (Kos and Ramadan 1977).

Noise at the upper end of the frequency range can be reduced to negligible proportions using acoustic filters. For frequencies below $1\ s^{-1}$ one may have to resort to the isolation technique, recently developed by Faller *et al* (1978) using the concept of a super spring, which is reported to be capable of isolating a system from mechanical noise of frequencies above $10^{-3}\ s^{-1}$.

Noise sources from other oscillation modes of the system (rocking modes, bulk oscillations) and flux creep in the cylinders have not been considered.

The entire detection scheme—including amplitude of noise signals, Q factor, resonant frequency, etc—can be tested by measuring the response of the detector to known energy sources. The spatial relationship between the tuning coil and the cylinders permits one to calibrate the detector by measuring the signal amplitude from the sensor coils at the resonant frequency to current pulses of predetermined amplitude in the tuning coil.

8. Conclusions

The design of a gravitational radiation detector with some unique and versatile features has been described. Order-of-magnitude estimates for the resonant frequencies as well as the correspdnding flux sensitivities have been derived.

The calculated sensitivities are comparable to those reported for existing detectors which are only sensitive to a narrow line in the expected spectrum of gravitational radiation. Sensitivities may be improved considerably if (i) high-Q materials are used, (ii) SQUID noise is reduced, and (iii) one has access to temperatures below 1 K.

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